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DETERMINATION OF STATIONARY TRAVELING WAVES ON NONLINEAR TRANSMISSION LINES

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ABSTRACT

This work describes how stationary waves traveling on nonlinear transmission lines may be determined within the framework of harmonic balance. A particular property of the given formulation is that sparse matrix packages optimized for the solution of harmonic balance Newton update equations are not disturbed. The method is demonstrated for an LC line which supports the propagation of solitons of a well-known, analytically given form.

INTRODUCTION

This work deals with technical details of the solution of the harmonic balance (HB) update equation in connection with further conditions that are required in autonomous cases to fix additional degrees of freedom. The term autonomous is used here in both a temporal and a spatial respect; it subsumes circuits that generate oscillations (including the case of forced oscillators) as well as circuits that are capable of guiding traveling waves with unknown shape and velocity, e.g. nonlinear lines. In the latter case of wave guiding structures the real circuit will not be autonomous in the strict sense (i.e. state equations are $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$, where $\mathbf{f}$ does not explicitly depend on t), as every traveling wave will have to be excited once, however, the part of the line that will be used to determine the wave will be.

SOLUTION OF THE UPDATE EQUATION

The solution procedure for the update equation will be given for the general case of several unknown fundamental frequencies. The procedure will then be applied to the special case of traveling waves on nonlinear lines, where one unknown frequency, given as a delay time, is sought.

Consider a network in a quasi-periodic electrical regime where voltages and currents may be given in the form

$$u_m(t) = \sum_n U_{m,n} \exp(j\omega_n t), \tag{1}$$

where $u_m(t)$ is the m-th unknown voltage or current, and the summation ranges over a set of frequencies generated by the intermodulation of N incommensurable fundamental tones $\omega_{f,i}$. A more detailed discussion of such conditions can be found in [1].

Suppose M fundamental frequencies are free. Introducing tuning factors $x_i, i = 1 \ldots M$, such that during simulation $\omega_{f,i} = (1 + x_i)\omega_{0f,i}$, where $\omega_{0f,i}$ are starting estimates for the free fundamentals, yields the augmented harmonic balance equation

$$\mathbf{F}(\mathbf{U}; x_1, \ldots, x_M) = \mathbf{0} \tag{2}$$

for the final solution vector $\mathbf{U}$ and the unknown tuning factors x_i. Due to the addition of M unknowns M further equations are required; commonly the phases of M suitable intermodulation products U_{k_i,h_i} (called "reference harmonics" in [1]) are set to zero. In this work we assume that the phases of these frequencies are to assume prescribed values $\phi_i, i = 1 \ldots M$. The resulting nonlinear system of equations is

$$\begin{aligned} \mathbf{F}(\mathbf{U}; x_1, \ldots, x_M) &= \mathbf{0} \\ U''_{k_1,h_1} - U'_{k_1,h_1} \tan\phi_1 &= 0 \\ &\vdots \\ U''_{k_M,h_M} - U'_{k_M,h_M} \tan\phi_M &= 0, \end{aligned} \tag{3}$$

where single and double primes denote real and imaginary parts, respectively.

A common means of solving HB equations is the Newton method. This requires the solution of linear systems of equations involving the Jacobian matrix of Eq. (3) which is preferably accomplished by LU decomposition of the Jacobian. One advantage of LU decomposition is that a decomposed Jacobian may be used for several steps, provided the Jacobian undergoes only minor changes from step to step, thus reducing computational costs. Usually the Jacobian is considerably sparse, according to the network topology, and is made up of a pattern of square and diagonal blocks the size of which corresponds to the number of considered frequencies. Therefore, the sparsity occurs on an intermediate level corresponding to the network topology, while inside blocks most places are non-zero unless (in the usual case where the state vector contains positive frequency components only) two components $G_{r,i+j}$ and $G_{r,i-j}$ of the underlying derivative vanish simultaneously. Specialized sparse matrix packages for HB applications take account of these properties to increase efficiency [2]. However, then, including a few phase equations as in Eq. (3) is somewhat cumbersome, because these equations do not match the general structure of the pure HB update equation corresponding to Eq. (2) with x_i held constant. In order to maintain efficiency, the update equation for Eq. (3) may be solved as shown below.

Consider the variation of the HB error for given variations of the state vector $\delta\mathbf{U}$ and the tuning factors δx_i,

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i.e.

$$\delta \mathbf{F} = \mathbf{J}\delta \mathbf{U} + \sum_i \mathbf{J}_i \delta x_i, \tag{4}$$

where $\mathbf{J}$ is the Jacobian of $\mathbf{F}$ with respect to $\mathbf{U}$ and $\mathbf{J}_i$ is the gradient of $\mathbf{F}$ with respect to x_i. The $\mathbf{J}_i$ may be determined e.g. by a straightforward modification of the equations given in [1].

Multiplying Eq. (4) with $\mathbf{J}^{-1}$ yields

$$\mathbf{R} \stackrel{def}{=} \mathbf{J}^{-1}\delta \mathbf{F} = \delta \mathbf{U} + \sum_i \mathbf{J}^{-1}\mathbf{J}_i \delta x_i \tag{5}$$

or

$$\mathbf{R} = \delta \mathbf{U} + \sum_i \mathbf{B}_i \delta x_i, \tag{6}$$

where $\mathbf{B}_i$ is the vector obtained by solving $\mathbf{J}\mathbf{B}_i = \mathbf{J}_i$. Once an LU decomposition of the actual Jacobian has been done, the $\mathbf{B}_i$ may be determined by computationally little expensive forward and backward substitutions. Given a prescribed value $\delta\hat{\mathbf{F}}$ (e.g. the last HB error multiplied by a negative relaxation factor), $\mathbf{R} = \mathbf{J}^{-1}\delta\hat{\mathbf{F}}$ is the change that U had to undergo if the x_i were fixed to constant values. Introducing the phase errors

$$p_i = U''_{k_i,h_i} - U'_{k_i,h_i} \tan \phi_i, \tag{7}$$

the update equation for Eq. (3) now writes as

$$\begin{aligned} \mathbf{R} &= \delta \mathbf{U} + \sum_i \mathbf{B}_i \delta x_i \\ \delta p_i &= \delta U''_{k_i,h_i} - \delta U'_{k_i,h_i} \tan \phi_i, \quad i = 1 \ldots M, \end{aligned} \tag{8}$$

where the δp_i are the prescribed changes in phase error for the actual step.

Extracting those equations from Eq. (8) involving the "reference harmonics" $\delta U_{k_i,h_i}$ yields a real linear system of equations of order $3M$ for the $3M$ quantities $\delta U'_{k_i,h_i}, \delta U''_{k_i,h_i}$, and $\delta x_i, i = 1 \ldots M$, which may be solved. The state vector update $\delta \mathbf{U}$ is then given by

$$\delta \mathbf{U} = \mathbf{R} - \sum_i \mathbf{B}_i \delta x_i. \tag{9}$$

For $M = 2$, letting $(k,h) = (k_1,h_1), (l,m) = (k_2,h_2), x = x_1, y = x_2, \phi = \phi_1, \psi = \phi_2, p = p_1$, and $q = p_2$, the above procedure leads to

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = N^{-1} \begin{pmatrix} R''_{k,h} - R'_{k,h} \tan \phi - \delta p \\ R''_{l,m} - R'_{l,m} \tan \psi - \delta q \end{pmatrix} \tag{10}$$

with

$$N = \begin{pmatrix} B''_{k,h} - B'_{k,h} \tan \phi & C''_{k,h} - C'_{k,h} \tan \phi \\ B''_{l,m} - B'_{l,m} \tan \psi & C''_{l,m} - C'_{l,m} \tan \psi \end{pmatrix}. \tag{11}$$

Setting $\phi = \psi = 0$ yields

$$\delta x = \frac{C''_{l,m}(R''_{k,h} - \delta p) - C''_{k,h}(R''_{l,m} - \delta q)}{B''_{k,h}C''_{l,m} - B''_{l,m}C''_{k,h}} \tag{12}$$

$$\delta y = \frac{B''_{k,h}(R''_{l,m} - \delta q) - B''_{l,m}(R''_{k,h} - \delta p)}{B''_{k,h}C''_{l,m} - B''_{l,m}C''_{k,h}}. \tag{13}$$

Similarly, for $M = 1$,

$$\delta x = \frac{R''_{k,h} - R'_{k,h} \tan \phi - \delta p}{B''_{k,h} - B'_{k,h} \tan \phi} \tag{14}$$

reduces to

$$\delta x = \frac{R''_{k,h} - \delta p}{B''_{k,h}} \tag{15}$$

for $\tan \phi = 0$.

For $M > 1$, harmonic balance has to be applied with caution because circuits supporting more than one free frequency are highly likely to exhibit complicated dynamic behaviour, e.g. erratic changes between infinitely many instable quasiperiodic states (the "quasi-periodic" route to chaos, cf. [3]), in which case waveforms can not be described by Eq. (1), though the HB solution process may seem to converge for low accuracy thresholds occasionally. However, there are cases with more than one free frequency which may succesfully be treated by HB, e.g. the mutual pulling of two weakly coupled, different oscillators.

STATIONARY TRAVELING WAVES

As far as the solution of the HB update equation is concerned, the tuning factors x_i are not explicitly considered as part of the state vector rather than as parameters that are used in conjunction with additional conditions to correct a "raw" update $\mathbf{R} = \mathbf{J}^{-1}\delta\hat{\mathbf{F}}$ to its final value, without disturbing sparse matrix solvers that are tailor-made for the particular properties of HB-Jacobians. However, the procedure is algebraically equivalent to the mixed-mode Newton iteration [1] if the Jacobian is regular.

A procedure similar to that above may be employed to determine stationary traveling waves on nonlinear lines by HB. Rather than fundamental frequencies, which are now given and constant, the shape and velocity of waves traveling on a homogeneous lossless nonlinear line of infinite length are sought. Consider a segment cut out of the line, with port quantities $v_i(t), i_i(t), i = 1, 2$, as shown in Fig. 1. For a traveling wave that is subject to a time delay τ without change of shape while traveling through the segment, it is required that

$$\begin{aligned} V_{1,n} &= \alpha \exp(j\omega_n \tau) V_{2,n} \\ I_{1,n} &= \alpha \exp(j\omega_n \tau) I_{2,n} \end{aligned} \tag{16}$$

for all Fourier components considered. Both α and τ are unknown. It should be noted that the determination of α and τ and the port quantities $V_{1,n}, I_{1,n}$ (plus state variables of the line model) is an autonomous problem with fundamental frequencies being fixed.

α is included for the sake of generality only, however, for waves traveling on lossless nonlinear lines the simulation will have to yield $\alpha = 1$ with high accuracy; a failure of this condition will indicate the presence of losses and thus the solution will not represent a wave traveling without change of shape. Waves traveling on lossy nonlinear lines will in general not only experience changes in "amplitude," but in overall shape.

For a properly posed problem, either one complex or two real additional conditions are required, because there are two additional real unknowns. Results of theoretical and experimental investigations [4, 5, 6] suggest that prescribing a Fourier component might serve this purpose. Therefore, without loss of generality, the condition

$$p \stackrel{def}{=} V_{1,1} - \hat{V}_{1,1} = 0 \tag{17}$$

for a given $\hat{V}_{1,1}$ will be imposed.

Eq. (16) may be considered to describe an imaginary device that may be included by standard HB means; the current leaving port 2, say, will have to be added as new state variable $\mathbf{I}$, if there is not already an equivalent quantity due to the segment model part of the state vector. The update equation is now

$$\begin{aligned} \delta\hat{\mathbf{F}} &= \mathbf{J}\delta\mathbf{U} + \mathbf{J}_\alpha\delta\alpha + \mathbf{J}_\tau\delta\tau \\ \delta p' &= \delta V'_{1,1} \\ \delta p'' &= \delta V''_{1,1}. \end{aligned} \tag{18}$$

If $\mathbf{F}_I$ is the part of the error vector corresponding to $\mathbf{I}$, i.e.

$$F_{I,n} = V_{1,n} - \alpha\exp(j\omega_n\tau)V_{2,n}, \tag{19}$$

the only non-zero elements of $\mathbf{J}_\alpha$ and $\mathbf{J}_\tau$ are

$$\frac{\partial F_{1,n}}{\partial\alpha} = -\exp(j\omega_n\tau)I_n \tag{20}$$

$$\frac{\partial F_{I,n}}{\partial\alpha} = -\exp(j\omega_n\tau)V_{2,n}, \tag{21}$$

and

$$\frac{\partial F_{1,n}}{\partial\tau} = -j\alpha\omega_n\exp(j\omega_n\tau)I_n \tag{22}$$

$$\frac{\partial F_{I,n}}{\partial\tau} = -j\alpha\omega_n\exp(j\omega_n\tau)V_{2,n}, \tag{23}$$

where currents leaving a node are taken to be positive.

Application of the above given procedure then yields

$$\delta\alpha = \frac{C''_{1,1}(R'_{1,1} - \delta p') - C'_{1,1}(R''_{1,1} - \delta p'')}{B'_{1,1}C''_{1,1} - B''_{1,1}C'_{1,1}} \tag{24}$$

$$\delta\tau = \frac{B'_{1,1}(R''_{1,1} - \delta p'') - B''_{1,1}(R'_{1,1} - \delta p')}{B'_{1,1}C''_{1,1} - B''_{1,1}C'_{1,1}} \tag{25}$$

with $\mathbf{B} = \mathbf{J}^{-1}\mathbf{J}_\alpha$, $\mathbf{C} = \mathbf{J}^{-1}\mathbf{J}_\tau$, and $\mathbf{R} = \mathbf{J}^{-1}\delta\hat{\mathbf{F}}$.

It should be noted that a priori there is not much more known about solutions that have been found by the above procedure, than that Eq. (16) is fulfilled to some degree of accuracy. There is especially no evidence whether such a solution represents a solitary wave or even a soliton (in the sense as given in [4]).

RESULTS

As a test for the method, a nonlinear transmission line has been chosen which supports the propagation of solitons the analytical form of which is known. Fig. 2(a) shows a segment of a nonlinear LC line with a voltage dependent capacitance $C(V)$. For the case $C(V) = C_0/(1 + V/V_0)$, a solution for a soliton is given by [6]

$$v_n(t) = V_{max}\,\mathrm{sech}^2\left(\sqrt{\frac{V_{max}}{LC_0V_0}}(t - nT_D)\right), \tag{26}$$

with

$$T_D = \sqrt{\frac{LC_0V_0}{V_{max}}}\sinh^{-1}\left(\sqrt{\frac{V_{max}}{V_0}}\right), \tag{27}$$

where $v_n(t)$ is the voltage at the n-th segment. A comparison of a numerical solution achieved by HB and the analytical solution for $L = 195.22\text{pH}, C_0 = 15.8445\text{fF}, V_0 = 0.65\text{V}$ and $V_{max} = 130.67\text{mV}$ is shown in Fig. 2(b) for $v_1(t)$ and $v_2(t)$. Corresponding curves of the numerical and analytical solution coincide within drawing accuracy. The HB result for α was $1 - 3 \cdot 10^{-14}$, and $T_D = 1.7045\text{ps}$ or 6.136°@10GHz, giving a good agreement with the value achieved by Eq. (27), $T_D = 1.7046\text{ps}$.

Fig. 3 shows the absolute error of the harmonic balance solution. There is a small DC offset visible as well as an alternating error due to the finite number of considered harmonics.

CONCLUSION

It has been shown how the determination of stationary traveling waves on nonlinear transmission lines may be achieved by harmonic balance. The method does not disturb specialized sparse matrix solvers that exploit particular properties of HB Jacobians. A comparison with an analytically given solution for a soliton on a nonlinear LC line shows good agreement.

REFERENCES

[1] V. Rizzoli and A. Neri, "Harmonic-balance analysis of multitone autonomous nonlinear microwave circuits," in *1991 IEEE MTT-S Int. Microwave Symp. Dig.*, Boston, pp. 107–110, June 1991.

[2] V. Rizzoli, F. Mastri, F. Sgallari, and V. Frontini, "The exploitation of sparse-matrix techniques in conjunction with the piecewise harmonic-balance method for nonlinear microwave circuit analysis," in *1990 IEEE MTT-S Int. Microwave Symp. Dig.*, Dallas, pp. 1295–1298, June 1990.

[3] J. A. Glazier and A. Libchaber, "Quasi-periodicity and dynamical systems: An experimentalist's view," *IEEE Trans. Circuits Syst.*, vol. CAS-35, pp. 790–809, July 1988.

[4] A. C. Scott, F. Y. F. Chu, and D. McLaughlin, "The soliton: A new concept in applied science," *Proc. IEEE*, vol. 61, pp. 1443–1483, Oct. 1973.

[5] R. Hirota and K. Suzuki, "Theoretical and experimental studies of lattice solitons in nonlinear lumped networks," *Proc. IEEE*, vol. 61, pp. 1483–1491, Oct. 1973.

[6] M. Case, M. Kamegawa, R. Yu, and M. J. W. Rodwell, "Impulse compression using soliton effects in a monolithic GaAs circuit," *Appl. Phys. Lett.*, vol. 58, pp. 173–175, Jan. 1991.

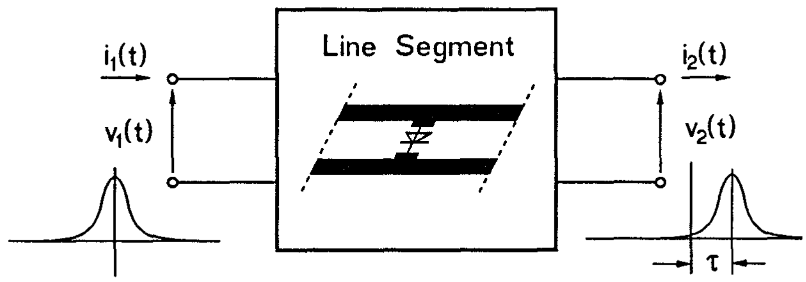

Fig. 1: A segment cut out of a line. The line is made up of cascaded identical segments. A stationary traveling wave experiences a time delay τ while running through the segment without change of shape.

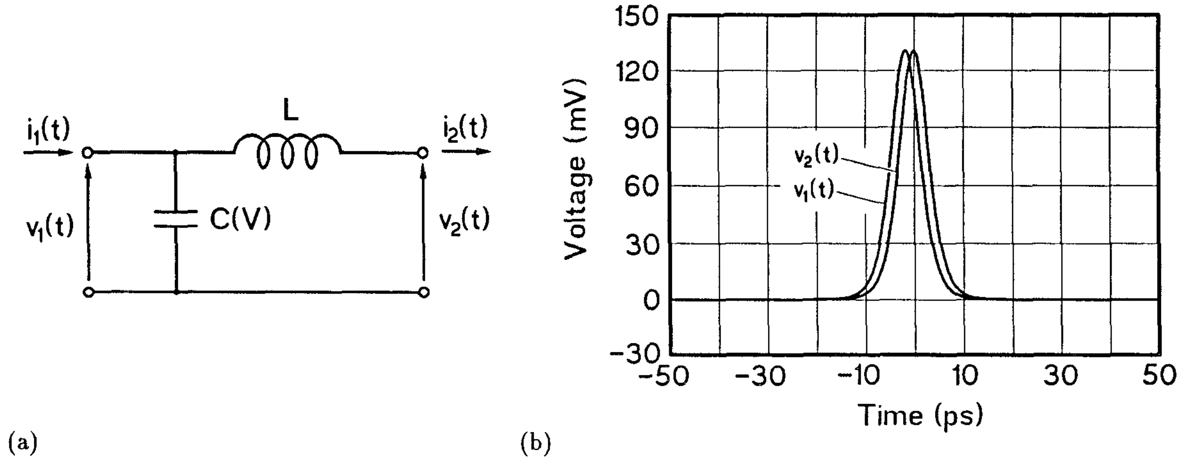

Fig. 2: (a) Segment of a nonlinear LC line. (b) Numerical (HB) and analytical solutions of $v_1(t)$ and $v_2(t)$ for the nonlinear LC line for the case $C(V) = C_0/(1+V/V_0)$ with $L = 195.22\text{pH}, C_0 = 15.8445\text{fF}, V_0 = 0.65\text{V}$, and $V_{max} = 130.67\text{mV}$. Within drawing accuracy, the numerical and the analytical results coincide for both $v_1(t)$ and $v_2(t)$. The HB calculations were performed with 30 harmonics.

Fig. 3: Absolute error of HB solution.